# POLARIMETRIC MICROWAVE INVERSE SCATTERING AS APPLIED TO NONDESTRUCTIVE TESTING

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## INTRODUCTION

The conventional procedures of nondestructive evaluation by means of ultrasonic techniques are limited by surface roughness and grain-boundary scattering of testmaterials. Especially ceramics must be coupled carefully with high-frequency ultrasonic transducers. Concerning these restrictions and in view of the possibility of contactless measurements of low-loss dielectric materials with microwaves, the application of electromagnetic waves in NDE appears to be a favourable alternative to conventional ultrasonic techniques [1].

During the past the utilization of electromagnetic waves in NDE seemed to be not very promising because of an inadequate resolution of flaws and pores, caused on one hand on too long wavelengths with respect to ascertainable material defects and on the other hand there was no suitable inversion algorithm available for electromagnetic waves. Providing microwave measurement stations for high-frequencies (200-GHz or more) we have to consider the applications of microwaves in NDE from new points of view.

Electromagnetic waves are considerably different from acoustic waves in their physical propagation properties and mathematical formulation. If we take arbitrary polarizations of electromagnetic waves into account, a vector formulation of the electromagnetic inverse scattering problem will be necessary. In this paper we want to develop a unified theory for the electromagnetic inverse scattering problem relying on the weak scattering (Born) approximation with full use of polarization information. This theory based on the scalar formulation of the multidimensional linearized scattering problem [2, 3] and follows [4, 5, 6], which essentially treat the electromagnetic inverse scattering problem within the physical optics (Kirchhoff) approximation.

## ELECTROMAGNETIC SCATTERING AND HOLOGRAPHIC FIELD REPRESENTATION

Fig. 1 shows the inverse scattering problem which has to be solved. From the scattered field data $\mathbf{E}_s$ which are known on the dashed surface $S_M$ (see Fig. 1) by measurements, we want to conclude about the geometry of the scatterer $V_c$. $\mathbf{E}_i$ is the electromagnetic incident field and $\mathbf{E}$ the total field anywhere in space, consisting of the incident and the scattered field. The reconstruction volume $V_c$ is enclosed by the arbitrarily closed measurement surface $S_M$.

With the aid of the free space dyadic Green function of the electromagnetic differential

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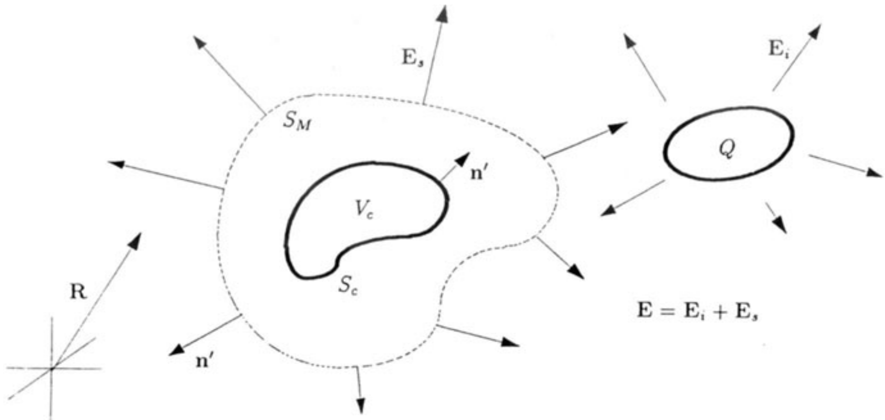


Fig. 1. Representation of a multidimensional electromagnetic direct or inverse scattering problem

equation in the frequency domain, according to

$$\mathbf{G}(\mathbf{R}-\mathbf{R}',\omega) = (\mathbf{I} + \frac{1}{k^2}\nabla\nabla)\frac{e^{jk|\mathbf{R}-\mathbf{R}'|}}{4\pi|\mathbf{R}-\mathbf{R}'|}\,, \tag{1}$$

we can determine the electromagnetic scattered field of the whole space outside the measurement surface $S_M$ with only knowledge of the **E**- and **H**-field on this surface. So the scattered field can be represented for the exterior space through the Stratton-Chu integral representation

$$\begin{aligned}\mathbf{E}_s(\mathbf{R},\omega) \;=\; & \iint_{S_M} \{j\omega\mu[\mathbf{n}' \times \mathbf{H}_s(\mathbf{R}',\omega)]\cdot\mathbf{G}(\mathbf{R}-\mathbf{R}',\omega) + \\ &+\; [\mathbf{n}' \times \mathbf{E}_s(\mathbf{R}',\omega)]\cdot\nabla' \times \mathbf{G}(\mathbf{R}-\mathbf{R}',\omega)\}dS'\,. \qquad \mathbf{R} \notin V_M \cup S_M\end{aligned} \tag{2}$$

For the interior of the measurement surface this equation only yields

$$\mathbf{E}_s(\mathbf{R},\omega) = -\mathbf{E}_i(\mathbf{R},\omega)\,, \qquad \mathbf{R} \in V_M \tag{3}$$

so that the total field is $\mathbf{E}(\mathbf{R},\omega) = 0$ for $\mathbf{R} \in V_M$. As a matter of common knowledge, this corresponds to the statement, heuristically found by Huygens, that a wavefront of a wavefield can be followed through pointwise propagation and subsequently forming its envelope. If we want to make any statements about the inner field values of $S_M$, we have to introduce a new definition of equ. (2). Replacing the dyadic Green function **G** by the complex conjugate one, we vividly backpropagate the point sources which are known with their amplitude distributions $\mathbf{E}_s$ and $\mathbf{H}_s$ on the measurement surface $S_M$. Therefore, the definition of the electromagnetic vector holographic field can be expressed by

$$\begin{aligned}\Theta_H^{\mathbf{E}}(\mathbf{R},\omega) \;=\; & -\iint_{S_M} \{j\omega\mu[\mathbf{n}' \times \mathbf{H}_s(\mathbf{R}',\omega)]\cdot\mathbf{G}^*(\mathbf{R}-\mathbf{R}',\omega) + \\ &+\; [\mathbf{n}' \times \mathbf{E}_s(\mathbf{R}',\omega)]\cdot\nabla' \times \mathbf{G}^*(\mathbf{R}-\mathbf{R}',\omega)\}\,dS'\,. \qquad \mathbf{R} \in V_M\end{aligned} \tag{4}$$

With introduction of secondary equivalent sources on the scatterer, which can be represented through the inhomogeneity of the differential equation for the electromagnetic field

$$\nabla \times \nabla \times \mathbf{E}_s - k^2\mathbf{E}_s = j\omega\mu\mathbf{J}_c \tag{5}$$

and with application of the vector Green theorem [7] on the vector holographic field (4), we obtain equ. (6), according to

$$\Theta_H^{\mathbf{E}}(\mathbf{R},\omega) = -2\omega\mu \iiint_{-\infty}^{+\infty} \mathbf{J}_c(\mathbf{R}',\omega)\cdot\mathbf{G}_i(\mathbf{R}-\mathbf{R}',\omega)\,d^3\mathbf{R}'\,, \tag{6}$$

as an extension of the scalar Porter-Bojarski integral equation of the electromagnetic field. $\mathbf{G}_i(\mathbf{R}-\mathbf{R}',\omega)$ represents the imaginary part of the dyadic Green function (1).

For the dielectric case of an isotropic scatterer we can express the equivalent sources through

$$j\omega\mu\mathbf{J}_c(\mathbf{R},\omega) = -k^2\,\underbrace{\Gamma(\mathbf{R})\left[1-\frac{k_c^2(\mathbf{R})}{k^2}\right]}_{=\,O(\mathbf{R})}\,\mathbf{E}(\mathbf{R},\omega) \tag{7}$$

with $k_c(\mathbf{R})$ as the wavenumber of the interior of the dielectric scatterer and $O(\mathbf{R})$ as the so-called object function.

## RECONSTRUCTION OF DIELECTRIC SCATTERERS

For weak scatterers we can introduce the Born approximation. The fundamental principle of the Born approximation is the disregard of the scattered field $\mathbf{E}_s$, $\mathbf{H}_s$ in the interior of the scatterer, so that we can state equ. (7) with only consideration of an incident field, according to

$$j\omega\mu\mathbf{J}_c^{\mathrm{Born}}(\mathbf{R},\omega) = -k^2 O(\mathbf{R})\hat{\mathbf{E}}_0\,F(\omega)e^{jk\hat{\mathbf{k}}_i\cdot\mathbf{R}}\,. \tag{8}$$

The incident field is chosen as a plane wave. Inserting equ. (8) in the electromagnetic Porter-Bojarski equation (6), we obtain for the holographic field

$$\Theta_H^{\mathbf{E}}(\mathbf{R},\omega) = -2jk^2F(\omega)\hat{\mathbf{E}}_0\cdot\iiint_{-\infty}^{+\infty}\mathbf{G}_i(\mathbf{R}-\mathbf{R}',\omega)\,O(\mathbf{R}')e^{jk\hat{\mathbf{k}}_i\cdot\mathbf{R}'}d^3\mathbf{R}'\,. \tag{9}$$

Now, we have to recast this equation for the object function. In order to do so, it is necessary to transform equ. (9) to the Fourier space. After multiplying this equation with $e^{-jk\hat{\mathbf{k}}_i\cdot\mathbf{R}}$ and subsequently, employing the threedimensional Fourier transform, we get

$$\tilde{\Theta}_H^{\mathbf{E}}(\mathbf{K}+k\hat{\mathbf{k}}_i,\omega) = -2jk^2F(\omega)\hat{\mathbf{E}}_0\cdot\left[\mathbf{G}_i(\mathbf{K}+k\hat{\mathbf{k}}_i,\omega)\tilde{O}(\mathbf{K})\right] \tag{10}$$

with

$$\mathbf{G}_i(\mathbf{K}+k\hat{\mathbf{k}}_i,\omega) = \frac{\pi}{2k}(\mathbf{I}-\frac{1}{k^2}(\mathbf{K}+k\hat{\mathbf{k}}_i)(\mathbf{K}+k\hat{\mathbf{k}}_i))\,\delta(|\mathbf{K}+k\hat{\mathbf{k}}_i|-k) \tag{11}$$

for $k>0$. Taking the scalar product of equ. (8) and $(\mathbf{K}\times\hat{\mathbf{k}}_i)$ and applying the relation $\hat{\mathbf{k}}_i\times\hat{\mathbf{E}}_0 = -\hat{\mathbf{E}}_{\mathrm{orth}}$, we can express equ. (10) as follows:

$$\frac{1}{k^2F(\omega)}(\mathbf{K}\times\hat{\mathbf{k}}_i)\cdot\tilde{\Theta}_H^{\mathbf{E}}(\mathbf{K}+k\hat{\mathbf{k}}_i,\omega) = j\frac{\pi}{k}\left[(\hat{\mathbf{E}}_{\mathrm{orth}}\cdot\mathbf{K})\tilde{O}(\mathbf{K})\right]\,\delta(|\mathbf{K}+k\hat{\mathbf{k}}_i|-k)\,. \tag{12}$$

Recognizing that equ. (12) is singular for $k=|\mathbf{K}+k\hat{\mathbf{k}}_i|$ because of the singularity of the $\delta$-distribution, we have to integrate over all frequencies to eliminate this singularity (frequency diversity mode), so that we can recast this equation for the object function in Fourier space. It is only permitted to integrate the wavenumber $k$ over the interval $[+\infty,0]$, since the threedimensional Fourier space spectrum of the imaginary part of the dyadic Green function is considered only for positiv wavenumbers $k$. Therefore, applying the general relation

$$\int_{-\infty}^{+\infty}\Phi(k)\delta(g(k))\,dk = \sum_{j=1}^{m}\frac{\Phi(k_j)}{|g'(k_j)|} \tag{13}$$

which can be utilized in calculating equ. (14)

$$\int_0^{+\infty}\frac{1}{k}\delta(|\mathbf{K}+k\hat{\mathbf{k}}_i|-k)\,dk = -\frac{u(-\mathbf{K}\cdot\hat{\mathbf{k}}_i)}{\mathbf{K}\cdot\hat{\mathbf{k}}_i}\,, \tag{14}$$

we obtain the object function in threedimensional Fourier space with the help of equ. (14), according to

$$\tilde{O}(\mathbf{K})u(-\mathbf{K}\cdot\hat{\mathbf{k}}_i) = \frac{1}{j\pi}\int_0^{+\infty} \frac{1}{k^2F(\omega)}\frac{\hat{\mathbf{k}}_i\cdot\mathbf{K}}{\hat{\mathbf{E}}_{\mathrm{orth}}\cdot\mathbf{K}}\left[\mathbf{K}\times\tilde{\Theta}_H^{\mathbf{E}}(\mathbf{K}+k\hat{\mathbf{k}}_i,\omega)\right]\cdot\hat{\mathbf{k}}_i\,dk\,. \tag{15}$$

By means of the inverse Fourier transform, we can exploit this formula with knowledge of the generalized vector holographic field in threedimensional Fourier space, so that it is possible to evaluate the object function in $\mathbf{R}$-space. With introduction of an abbreviation of the integrated vector holographic field, as stated in equ. (16)

$$\tilde{\mathbf{Y}}(\mathbf{K}) = \frac{1}{j\pi}\int_0^{+\infty}\frac{1}{k^2F(\omega)}\tilde{\Theta}_H^{\mathbf{E}}(\mathbf{K}+k\hat{\mathbf{k}}_i,\omega)\,dk \tag{16}$$

and application of the threedimensional inverse Fourier transform, we can formally write the object function in $\mathbf{R}$-space:

$$O(\mathbf{R}) = \hat{\mathbf{k}}_i\cdot\mathcal{F}_{3\mathrm{D}}^{-1}\left\{\frac{\hat{\mathbf{k}}_i\cdot\mathbf{K}}{\hat{\mathbf{E}}_{\mathrm{orth}}\cdot\mathbf{K}}\left[\mathbf{K}\times\tilde{\mathbf{Y}}(\mathbf{K})\right]\right\}\,. \tag{17}$$

Here, the $\mathcal{F}_{3\mathrm{D}}^{-1}$-operator denotes the threedimensional inverse Fourier transformation.

A better and simpler method to evaluate equ. (15) can be formulated, if the generalized vector holographic field can be expressed in a suitable manner by the electromagnetic scattered field in $\mathbf{R}$-space on the measurement surface in the far-field. In order to do so, we have to take the threedimensional inverse Fourier transform with respect to the inserted abbreviation (16)

$$\mathcal{F}_{3\mathrm{D}}^{-1}\left\{\tilde{\mathbf{Y}}(\mathbf{K})\right\} = \mathbf{Y}(\mathbf{R}) = \frac{1}{j\pi}\int_0^{+\infty}\frac{1}{k^2F(\omega)}\Theta_H^{\mathbf{E}}(\mathbf{R},\omega)\,e^{-jk\hat{\mathbf{k}}_i\cdot\mathbf{R}}dk\,. \tag{18}$$

Formally, the generalized vector holographic field can be written as an integration with regard to the unit-sphere $S^2$, according to

$$\Theta_H^{\mathbf{E}}(\mathbf{R},\omega) = \frac{jk}{2\pi}\iint_{S^2}(\mathbf{I}-\hat{\mathbf{R}}'\hat{\mathbf{R}}')\cdot\mathbf{C}(\hat{\mathbf{R}}',\omega)\,e^{jk\hat{\mathbf{k}}_i\cdot\hat{\mathbf{R}}'}\,d^2\hat{\mathbf{R}}'\,, \tag{19}$$

in which the term $d^2\hat{\mathbf{R}}' = \frac{\mathbf{n}\cdot\mathbf{R}'}{\mathbf{R}'^2}dS_M$ denotes the differential solid angle element of equ. (19). $\mathbf{C}(\hat{\mathbf{R}}',\omega)$ is the scattering amplitude which only depends on frequency and the observation direction. It is contracted by the dyadic $(\mathbf{I}-\hat{\mathbf{R}}'\hat{\mathbf{R}}')$, so that its directivity is changed in magnitude and orientation. The radiation pattern is given by a volume integral over the equivalent secondary sources

$$\mathbf{C}(\hat{\mathbf{R}}',\omega) = \frac{j\omega\mu}{4\pi}\iiint_{-\infty}^{+\infty}\mathbf{J}_c(\mathbf{R}'',\omega)\,e^{-jk\hat{\mathbf{R}}'\cdot\mathbf{R}''}d^3\mathbf{R}'' \tag{20}$$

with $\hat{\mathbf{R}}'$ as a unit vector directed towards the measurement surface $S_M$. Fig. 2 shows the evaluation of the generalized vector holographic field for the equivalent secondary sources on the scatterer.

Relation (19) can be well applied, because the expression $(\mathbf{I}-\hat{\mathbf{R}}'\hat{\mathbf{R}}')\cdot\mathbf{C}(\hat{\mathbf{R}}',\omega)$ even represents the experimental data on the surface $S_M$ in the far-field. Considering equ. (2) with assistance of the vector Green theorem, we obtain the electromagnetic scattered field as a volume integration over the equivalent sources $\mathbf{J}_c$

$$\mathbf{E}_s(\mathbf{R},\omega) = j\omega\mu\iiint_{-\infty}^{+\infty}\mathbf{J}_c(\mathbf{R}',\omega)\cdot\mathbf{G}(\mathbf{R}-\mathbf{R}',\omega)\,d^3\mathbf{R}'\,. \qquad \mathbf{R}\notin V_c\cup S_c \tag{21}$$

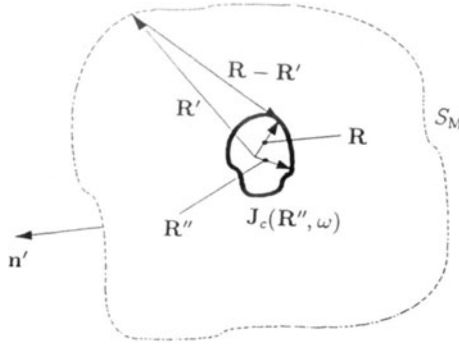


Fig. 2. Evaluation of the generalized vector holographic field for a measurement surface in the far-field

With introduction of the far-field approximation with $R$ greater than the dimension of the scatterer and $kR \gg 1$, it follows from equ. (21) with utilization of the dyadic Green function (1)

$$\mathbf{E}_s^{\text{far}}(\mathbf{R},\omega) = \frac{e^{jkR}}{R}(\mathbf{I} - \hat{\mathbf{R}}\hat{\mathbf{R}}) \cdot \mathbf{C}(\hat{\mathbf{R}},\omega) . \tag{22}$$

Now, inserting the generalized vector holographic field (19) in the abbreviated formulation (18) and subsequently, taking the threedimensional Fourier transform, we get a new expression for $\tilde{\mathbf{Y}}$ in $\mathbf{K}$-space

$$\tilde{\mathbf{Y}}(\mathbf{K}) = 4\pi \int\limits_0^{+\infty} \frac{1}{kF(\omega)} \iint_{S^2} (\mathbf{I} - \hat{\mathbf{R}}'\hat{\mathbf{R}}') \cdot \mathbf{C}(\hat{\mathbf{R}}',\omega)\, \delta[\mathbf{K} - k(\hat{\mathbf{R}}' - \hat{\mathbf{k}}_i)]\, d^2\hat{\mathbf{R}}' dk . \tag{23}$$

Note, that a $\delta$-distribution occurs in equ. (23), which can be used to transform the object function into $\mathbf{R}$-space. Inserting equ. (23) into the object representation in $\mathbf{K}$-space (equ. (15)), and considering the integrand of the resulting solid angle integral, we can utilize the special properties of the delta distribution, according to

$$\frac{\hat{\mathbf{k}}_i \cdot \mathbf{K}}{\hat{\mathbf{E}}_{\text{orth}} \cdot \mathbf{K}}(\hat{\mathbf{k}}_i \times \mathbf{K})\, \delta[\mathbf{K} - k(\hat{\mathbf{R}}' - \hat{\mathbf{k}}_i)] = \frac{1 - \hat{\mathbf{k}}_i \cdot \hat{\mathbf{R}}'}{\hat{\mathbf{E}}_{\text{orth}} \cdot \hat{\mathbf{R}}'}(\hat{\mathbf{R}}' \times k\hat{\mathbf{k}}_i)\, \delta[\mathbf{K} - k(\hat{\mathbf{R}}' - \hat{\mathbf{k}}_i)] . \tag{24}$$

Performing the threedimensional Fourier transform which has to be in the following form, because of the unit step function in equ. (15)

$$F(\mathbf{R}) = 2\Re\left\{ \frac{1}{(2\pi)^3} \iiint\limits_{-\infty}^{+\infty} \tilde{O}(\mathbf{K}) u(\mathbf{K} \cdot \hat{\mathbf{k}}_i)\, e^{j\mathbf{K}\cdot\mathbf{R}}\, d^3\mathbf{K} \right\} \tag{25}$$

with

$$\tilde{O}(\mathbf{K}) u(-\mathbf{K} \cdot \hat{\mathbf{k}}_i) = \tilde{F}(\mathbf{K}) \tag{26}$$

and employing the equation

$$\frac{1}{(2\pi)^3} \iiint\limits_{-\infty}^{+\infty} \delta(\mathbf{K} - k(\hat{\mathbf{R}}' - \hat{\mathbf{k}}_i))\, e^{j\mathbf{K}\cdot\mathbf{R}}\, d^3\mathbf{K} = e^{jk\mathbf{R}\cdot(\hat{\mathbf{R}}' - \hat{\mathbf{k}}_i)} , \tag{27}$$

we obtain the object function in $\mathbf{R}$-space

$$O(\mathbf{R}) = \frac{1}{2\pi^2}\Re\left\{ \int\limits_0^{+\infty} \frac{1}{F(\omega)} \iint_{S^2} \frac{1 - \hat{\mathbf{k}}_i \cdot \hat{\mathbf{R}}'}{\hat{\mathbf{E}}_{\text{orth}} \cdot \hat{\mathbf{R}}'}(\hat{\mathbf{R}}' \times \hat{\mathbf{k}}_i) \cdot (\mathbf{I} - \hat{\mathbf{R}}'\hat{\mathbf{R}}') \cdot \mathbf{C}(\hat{\mathbf{R}}',\omega) \right.$$
$$\left. e^{jk\hat{\mathbf{R}}'\cdot\mathbf{R}} d^2\hat{\mathbf{R}}'\, e^{-jk\hat{\mathbf{k}}_i\cdot\mathbf{R}} dk \right\} . \tag{28}$$

The experimental data $(\mathbf{I}-\hat{\mathbf{R}}'\hat{\mathbf{R}}')\cdot\mathbf{C}(\hat{\mathbf{R}}',\omega)$ in equ. (28) can be expressed by the electromagnetic far-field relation (equ. (22)), so we find the desired result inserting equ. (22) into equ. (28)

$$O(\mathbf{R}) = \frac{1}{2\pi^2}\Re\Bigg\{\int_0^{+\infty}\frac{1}{F(\omega)}\iint_{S^2}\frac{1-\hat{\mathbf{k}}_i\cdot\hat{\mathbf{R}}'}{\hat{\mathbf{E}}_{\text{orth}}\cdot\hat{\mathbf{R}}'}(\hat{\mathbf{R}}'\times\hat{\mathbf{k}}_i)\cdot\mathbf{E}_s^{\text{far}}(\mathbf{R}',\omega)$$

$$R'e^{-jkR'}e^{jk\hat{\mathbf{R}}'\cdot\mathbf{R}}d^2\hat{\mathbf{R}}'\,e^{jk\hat{\mathbf{k}}_i\cdot\mathbf{R}}dk\Bigg\}\,. \tag{29}$$

There are several methods available to evaluate equ. (28) and equ. (29) numerically, listed as follows.

Explicit numerical integration

As the first possibility of evaluation we consider the numerical integration. The solid angle integration in equ. (28) and equ. (29) can be expressed by introducing the spherical coordinates $\vartheta'$ and $\varphi'$ for the scalar and vector product, respectively, so that we can integrate over $\vartheta'$ and $\varphi'$ numerically. The numerical integration of the wavenumber $k$ must be cut off prematurely, because of its infinite upper integration limit. Hence, this causes numerical errors which become apparent in a reduced resolution of sharp contours. The advantage of this approach is that all reconstruction points of the scatterer can be calculated independently of each other. If only a cross-section of the scatterer is requested to be reconstructed, it is more effective to use an explicit numerical integration scheme.

Time domain backprojection

The second possible evaluation we want to consider is the time domain backprojection method which relies on the scalar case of [2]. For this purpose we have to transform the formula (28) in the time domain with respect to the frequency by means of the inverse Fourier transform, i.e. the $k$-integration of equ. (28) must be converted into an $\omega$-integration and the lower integration limit has to be expanded to $-\infty$. With the aid of the unit-step function $u(\omega)$ and the insertion of an $e^{-j\omega t}$-term for $t=0$, we can write equ. (28) as a Fourier integral with respect to $\omega$, according to

$$O(\mathbf{R}) = \frac{1}{c\pi}\Re\Bigg\{\iint_{S^2}\frac{1-\hat{\mathbf{k}}_i\cdot\hat{\mathbf{R}}'}{\hat{\mathbf{E}}_{\text{orth}}\cdot\hat{\mathbf{R}}'}(\hat{\mathbf{R}}'\times\hat{\mathbf{k}}_i)\cdot\frac{1}{2\pi}\int_{-\infty}^{+\infty}(\mathbf{I}-\hat{\mathbf{R}}'\hat{\mathbf{R}}')\cdot\mathbf{C}(\hat{\mathbf{R}}',\omega)$$

$$\frac{u(\omega)}{F(\omega)}e^{-j\omega t'}d\omega\,d^2\hat{\mathbf{R}}'\Bigg\}\Bigg|_{t=0} \tag{30}$$

with the substitution

$$t' = t - \frac{\mathbf{R}\cdot(\hat{\mathbf{R}}'-\hat{\mathbf{k}}_i)}{c}\,. \tag{31}$$

Only considering the frequency Fourier integral, we obtain with application of the convolution theorem

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}(\mathbf{I}-\hat{\mathbf{R}}'\hat{\mathbf{R}}')\cdot\mathbf{C}(\hat{\mathbf{R}}',\omega)\frac{u(\omega)}{F(\omega)}e^{-j\omega t'}d\omega =$$

$$\underbrace{\mathcal{F}^{-1}\left\{\frac{1}{F(\omega)}(\mathbf{I}-\hat{\mathbf{R}}'\hat{\mathbf{R}}')\cdot\mathbf{C}(\hat{\mathbf{R}}',\omega)\right\}}_{=\,\mathbf{E}_s^{\text{Born}}(\hat{\mathbf{R}}',t')}\star\mathcal{F}^{-1}\Big\{u(\omega)\Big\}\;. \tag{32}$$

Therefore, we readily get the formal time domain solution of the left convolution term of the right-hand side of equ. (32), considering the electromagnetic field within the far-field approximation. The right-hand side of equ. (32) yields a $\delta$-distribution as the real part of the inverse Fourier transform of a unit-step function. The imaginary part of this expression (equ. (33)) represents Cauchy's principal value of the Fourier integral of a sign-function [8] and exists only in a distributional sense. PV denotes the principal value of the integral. So we can write

$$\mathcal{F}^{-1}\Big\{u(\omega)\Big\} = \frac{1}{2}\delta(t') + \frac{1}{2\pi}PV\Big\{\frac{1}{jt'}\Big\} \ . \tag{33}$$

Concerning the fact, that only the real part of equ. (30) is used to evaluate this formula and that $\mathbf{E}_s^{\mathrm{Born}}(\hat{\mathbf{R}}', t)$ is real valued, we notice that the principal value of equ. (33) is not to be considered in further calculations. Inserting equ. (33) in (32) and equ. (32) in (30), we obtain in consideration of the substitution (31) the time domain backprojection formula

$$O(\mathbf{R}) = \frac{1}{2\pi c}\iint_{S^2} \mathbf{E}_s^{\mathrm{Born}}\left[\hat{\mathbf{R}}', t - \frac{\mathbf{R}\cdot(\hat{\mathbf{R}}' - \hat{\mathbf{k}}_i)}{c}\right]\Bigg|_{t=0} \cdot (\hat{\mathbf{R}}' \times \hat{\mathbf{k}}_i)\frac{1 - \hat{\mathbf{k}}_i \cdot \hat{\mathbf{R}}'}{\hat{\mathbf{E}}_{\mathrm{orth}} \cdot \hat{\mathbf{R}}'}\, d^2\hat{\mathbf{R}}' . \tag{34}$$

<u>Threedimensional Fourier inversion</u>

The third possibility in evaluating equ. (28) is the threedimensional Fourier transform. Equ. (28) can be formally transfered to a threedimensional Fourier integral. With definition of a Fourier space vector, according to

$$\mathbf{K} = k(\hat{\mathbf{R}}' - \hat{\mathbf{k}}_i) \tag{35}$$

and computation of the Jacobian

$$d^3\mathbf{K} = -k\hat{\mathbf{k}}_i \cdot \mathbf{K} d^2\hat{\mathbf{R}}' dk = k^2(1 - \hat{\mathbf{k}}_i \cdot \hat{\mathbf{R}}')\, d^2\hat{\mathbf{R}}' dk \tag{36}$$

we can express $O(\mathbf{R})$ by

$$O(\mathbf{R}) = 2\Re\Big\{\frac{1}{(2\pi)^3}\iiint_{-\infty}^{+\infty} \underbrace{\frac{2\pi}{k^2 F(\omega)}\frac{\hat{\mathbf{R}}' \times \hat{\mathbf{k}}_i}{\hat{\mathbf{R}}' \cdot \hat{\mathbf{E}}_{\mathrm{orth}}} \cdot (\mathbf{I} - \hat{\mathbf{R}}'\hat{\mathbf{R}}') \cdot \mathbf{C}(\hat{\mathbf{R}}', \omega)}_{= \tilde{O}(\mathbf{K})\, u(-\mathbf{K}\cdot\hat{\mathbf{k}}_i)}\, e^{j\mathbf{K}\cdot\mathbf{R}}\, d^3\mathbf{K}\Big\} \ . \tag{37}$$

A geometrical illustration of the definition of the $\mathbf{K}$-vector in Fourier space, according to equ. (35), is given in Fig. 3. Concerning this coordinate transform, we recognize that the variable $\hat{\mathbf{R}}' \in S^2$ and the wavenumber $k$ in the range $[0, +\infty]$ can only transform the $\mathbf{K}$-vector into a threedimensional Fourier half-space. The object function in this Fourier space can be read off immediately from the integrand of equ. (37) leaving one $\mathbf{K}$-half-space out of account, according to the unit-step function $u(-\mathbf{K}\cdot\hat{\mathbf{k}}_i)$. With insertion of the electromagnetic far-field term (equ. (22)) we get

$$\tilde{O}(\mathbf{K})u(-\mathbf{K}\cdot\hat{\mathbf{k}}_i) = \frac{\hat{\mathbf{R}}' \times \hat{\mathbf{k}}_i}{\hat{\mathbf{R}}' \cdot \hat{\mathbf{E}}_{\mathrm{orth}}} \cdot \frac{2\pi R'\, e^{-jkR'}}{k^2 F(\omega)} \mathbf{E}_s^{\mathrm{far}}(\mathbf{R}', \omega)\Big|_{\mathbf{K}=k(\hat{\mathbf{R}}' - \hat{\mathbf{k}}_i)} \ . \tag{38}$$

This equation relates the scattered data within the far-field approximation to the object function in $\mathbf{K}$-space, so that there are no explicit calculations of the generalized vector holographic field necessary. In conclusion, we obtain the solution of the object function in $\mathbf{R}$-space as a result of the real part of the numerical inverse Fourier transform.

As a final remark we can say that the main advantage of this approach is its speed compared with the explicit numerical integration if the whole scatterer is to be reconstructed.

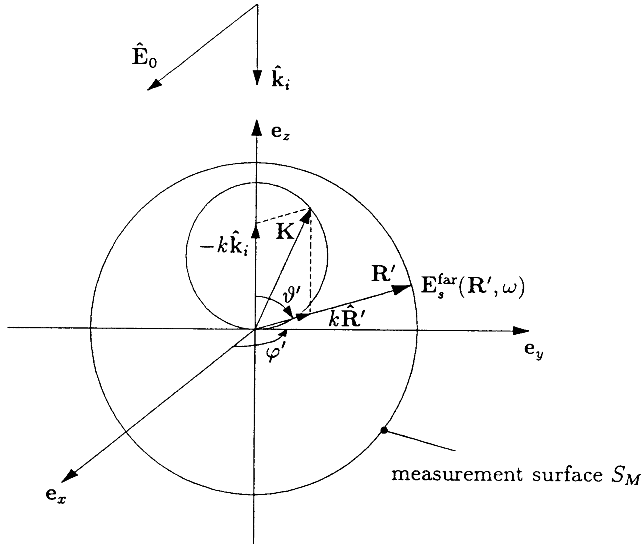


Fig. 3. Definition of a **K**-Vector in Fourier space in terms of the scattered field within the far-field approximation


REFERENCES

1 A.J. Bahr: *Nondestructive Microwave Evaluation of Ceramics*, IEEE Transactions on Microwave Theory and Techniques, MTT-26 (1978) 676

2 G.T. Herman, H.K. Tuy, K.J. Langenberg, P. Sabatier: *Basic Methods of Tomography and Inverse Problems*, Adam Hilger, Bristol 1987

3 K.J. Langenberg: *Introduction to the Special Issue on Inverse Problems*, Wave Motion 11 (1989) 99

4 M. Brandfaß: *Breitbandige polarimetrische Holographie mit elektromagnetischen Wellen am Beispiel einer idealleitenden Kugel*, M.S. Thesis of the University of Kassel, Dept. El. Eng., Kassel, FRG, 1989

5 K.J. Langenberg, M. Brandfaß, P. Fellinger, T. Gurke, T. Kreutter: *Polarimetrische Breitbandholographie mit Mikrowellen*, Kleinheubacher Berichte 34 (1991) 311

6 K.J. Langenberg, M. Brandfaß, P. Fellinger, T. Gurke, T. Kreutter: *A Unified Theory of Multidimensional Electromagnetic Vector Inverse Scattering within the Kirchhoff or Born Approximation*, in: Modern Problems in Radar Target Imaging (Ed.: H. Überall), Springer-Verlag, Berlin 1992 (to appear)

7 H.C. Chen, *Theory of Electromagnetic Waves*, McGraw-Hill, New York 1983

8 G. Doetsch: *Funktionaltransformationen in: Mathematische Hilfsmittel des Ingenieurs*, (Eds.: by R. Sauer and I. Szabó), Springer, Berlin 1967